Field Theory of Open and Closed Strings with Discrete Target Space

I. K. Kostov *◇

Service de Physique Théorique † *de Saclay*

CE-Saclay, F-91191 Gif-sur-Yvette, France

We study a $U(N)$-invariant vector+matrix chain with the color structure of a lattice gauge theory with quarks and interpret it as a theory of open and closed strings with target space $\mathbb{Z}$. The string field theory is constructed as a quasiclassical expansion for the Wilson loops and lines in this model. In a particular parametrization this is a theory of two scalar massless fields defined in the half-space $\{x \in \mathbb{Z}, \tau > 0\}$. The extra dimension τ is related to the longitudinal mode of the strings. The topology-changing string interactions are described by a local potential. The closed string interaction is nonzero only at boundary $\tau = 0$ while the open string interaction falls exponentially with τ.

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* on leave from the Institute for Nuclear Research and Nuclear Energy, 72 Boulevard Tsarigradsko Chaussée, 1784 Sofia, Bulgaria

◇ e-mail:kostov@amoco.saclay.cea.fr

† Laboratoire de la Direction des Sciences de la Matière du Commissariat à l'Energie Atomique

Introduction

The $D = 1$ closed string theory is known to describe a special critical behaviour of one-dimensional $N \times N$ matrix models. Remarkably, the discrete matrix chain leads to the same string theory as the continuum model (known as matrix quantum mechanics), under the condition that the lattice spacing Δ is smaller than some critical value, which we assume equal to 1. At $\Delta = 1$ the system undergoes a Kosterlitz-Thouless type transition and if $\Delta > 1$ the matrices decouple [1].

The appearance of a minimal length in the target space, anticipated by Klebanov and Susskind in [2], seems to be a fundamental property of the string theory. It signifies that the string theory has much fewer short-distance degrees of freedom than the conventional quantum field theory. As a consequence of this, the continuous target space can be restricted to a lattice $\mathbb{Z} \subset \mathbb{R}$ without loss of information[1].

The physics in the target space $\mathbb{Z}$ seems to be simpler than in $\mathbb{R}$. The loop amplitudes restricted to $\mathbb{Z}$ enjoy some nice factorization properties. Furthermore, as a consequence of the periodicity of the momentum space, an infinite set of "discrete" states with integer momenta become invisible in the space $\mathbb{Z}$. Therefore, it seems advantageous to consider a string theory on a lattice with spacing not inferiour but equal to the Kosterlitz-Thouless distance.

The string theory with target space $\mathbb{Z}$ has been originally constructed in [4] as an SOS model on a surface with fluctuating geometry. The secret of its solvability is the possibility to be mapped onto a gas of nonintersecting (but otherwise noninteracting) loops on the world sheet. The loops on the world sheet define a natural discretization of the moduli space and a possibility to construct an unambiguous string field diagram technique [5]. The loop gas approach can be easily generalized to to open strings. The critical behavior of the open strings with target space $\mathbb{Z}$ has been studied by the loop gas method in [6].

[1] It has been checked [3] that the n-loop tree-level amplitudes ($n \leq 4$) in the closed string theory with target space $\mathbb{R}$ can be reproduced from their restrictions in $\mathbb{Z}$.

To study of the topology-changing interactions of open and closed strings we need more advanced technology than the world sheet surgery applied in [5] and [6]. Such might be provided by an underlying large N field-theoretical model, as in the case of the continuum string theory.

In this letter we construct a one-dimensional lattice model with local $U(N)$ symmetry, whose color structure is that of a lattice gauge theory with quarks, and show that it is equivalent to the field theory of closed and open strings in $\mathbb{Z}$. The mean field problem in this model is the one-matrix integral with potential determined dynamically. The quasi-classical expansion for the Wilson loops and lines yields the string field Feynman rules. The vertices for the string fields have the geometrical interpretation of surfaces with various topologies localized at a single point of the target space.

In a special parametrization, the effective action is this of theory of scalar fields in the comb-like space $(x, \tau),\ x \in \mathbb{Z}, \tau > 0$. The kinetic term for these fields involves finite-difference operators in x and $i\tau$ directions. The interactions are described by a local nonpolynolial potential. The closed strings interact only along the boundary $\tau = 0$ while the coupling of the open strings falls exponentially in the bulk.

Closed and open strings from a $U(N)$ matrix-vector chain

The underlying lattice model possesses local $U(N)$ symmetry and resembles a Wilson lattice gauge theory, with the unitary measure for the "gluon" field replaced by a Gaussian measure. The Gaussian measure allows the eigenvalues of the gauge field to fluctuate in the radial direction, which leads to the longitudinal (Liouville) mode of the string. The vacuum energy of the model is equal to the partition function of a gas of triangulated surfaces with free boundaries, immersed in the lattice $\mathbb{Z}$.

The fluctuating variables associated with each point $x \in \mathbb{Z}$ are a fermion $\psi_x = \{\psi_x^i\}, \bar{\psi}_x = \{\bar{\psi}_x^i\}$, a hermitian matrix $\Phi_x = \Phi_x^\dagger = \{\Phi_x^{ij}\}$, and a complex matrix $A_x = \{A_x^{ij}\}$ with color indices ranging from 1 to N.

To simplify notations we combine the color index i and the space coordinate x into a double index $a = \{i, x\}, \quad i = 1, ..., N, \quad x \in \mathbb{Z}$. Then the entities of the model are the vector with anticommuting coordinates

$$\psi_a = \psi_{ix}, \quad \bar{\psi}_a = \bar{\psi}_{ix}, \tag{1}$$

and the hermitian matrix

$$A_{aa'} = \delta_{x,x'} \Phi_x^{ij} + \delta_{x,x'-1}\ A_x^{ij} + \delta_{x,x'+1}\ A_{x-1}^{\dagger\ ij}; \qquad a = \{i, x\},\ a' = \{j, x'\}. \tag{2}$$

The partition function is defined by the integral

$$\mathcal{Z} = \int [dA]\ d\bar{\psi} d\psi \exp\big[-\frac{1}{2}\mathrm{tr}A^2 + \frac{\lambda}{3\sqrt{N}}\mathrm{tr}A^3 - \bar{\psi}\psi + \lambda_B \bar{\psi} A \psi \big] \tag{3}$$

where the trace is understood in the sense of the double index a and $[dA]$ is the homogeneous measure for the nonzero matrix elements (2).

The perturbation series for the free energy $\mathcal{F} = \log \mathcal{Z}$ is a sum over connected "fat" graphs dual to triangulated surfaces with boundaries. The "windows" of the fat graph are spanned on the index lines labeled by double indices $a = \{i, x\}$. Therefore an integer coordinate x is assigned to each point of the triangulated surface. The free energy is equal to the sum of all connected surfaces $\mathcal{S}$ with free boundaries, immersed in $\mathbb{Z}$

$$\mathcal{F} = \sum_{\mathcal{S}} (-N)^{\chi} \lambda^S \lambda_B^{L_B} \tag{4}$$

where $\chi = 2 - 2\#(\text{handles}) - \#(\text{boundaries})$ is the Euler characteristics, $S = \#(\text{triangles})$ is the area, and $L_B = \#(\text{edges})$ is the total length of the boundaries of the surface $\mathcal{S}$. The gauge invariant operators creating closed and open strings are the Wilson loops and lines constructed in the same way as in the lattice gauge theory. We restrict ourselves to closed and open strings localized at a single point x

$$W_x(\ell) = \mathrm{tr}\ e^{\ell \Phi_x}, \quad \Omega_x(\ell) = \bar{\psi}_x e^{\ell \Phi_x} \psi_x \tag{5}$$

where the parameter ℓ is the (intrinsic) length of the string. Since the time slice of the one-dimensional spacetime consists of a single point, the operators (5) generate the whole Hilbert space. In this case the A-matrices are redundant variables and will be integrated out.

In the following we will consider a more general action containing source terms J and J^B. It is convenient to absorb the coupling constants λ and λ_B into the source and shift $\Phi_x \to \Phi_x + (2\lambda)^{-1} I$, where I is the unit matrix. Then, after performing the Gaussian integral over the A-variables, we find

$$\mathcal{Z}[J, J^B] = \int \prod_x d\psi_x e^{\mathrm{tr}\ J_x(\Phi_x)}\ d\bar{\psi}_x d\Phi_x e^{\bar{\psi}_x\ J^B_x(\Phi_x)\psi_x}\quad e^{\mathcal{W}} \tag{6}$$

$$\begin{aligned} \mathcal{W} = -\tfrac{1}{2} \sum_{x,x'} C_{xx'} \Big(& \log|\det(I \otimes \Phi_x + \Phi_{x'} \otimes I)| \\ & + [\psi_x \otimes \bar{\psi}_{x'}][I \otimes \Phi_x + \Phi_{x'} \otimes I]^{-1}[\psi_{x'} \otimes \bar{\psi}_x]\Big) \end{aligned} \tag{7}$$

where by $C_{xx'}$ we denoted the incidence matrix of the target space lattice $\mathbb{Z}$

$$C_{xx'} = \delta_{x,x'+1} + \delta_{x,x'-1}. \tag{8}$$

As a consequence of the local $U(N)$ symmetry the only relevant degrees of freedom are the N real eigenvalues ϕ_{ix} of the hermitian matrix Φ_x and the commuting nilpotent variables $\theta_{ix} = \psi_{ix}\bar{\psi}_{ix}$. The integration measure factorizes into the Haar measure in the $U(N)$ group and an integration measure along the radial directions ϕ_{ix}, θ_{ix}

$$d\Phi_x\ d\bar{\psi}_x d\psi_x = \text{constant} \times \prod_{i=1}^{N} d\phi_{ix} d\theta_{ix}\ \Delta^2(\phi_x) \tag{9}$$

where $\Delta(\phi)$ is the Vandermonde determinant

$$\Delta(\phi) = \prod_{i<j} (\phi_i - \phi_j). \tag{10}$$

The algebra and the integration over the θ-variables are defined by the rules

$$\theta\theta' = \theta'\theta, \quad \theta^2 = 0, \quad \int d\theta = 0, \int d\theta\ \theta = 1. \tag{11}$$

To save space we will use the following compact notations

$$\hat{\phi}_{ix} = \{\phi_{ix}, \theta_{ix}\}, \quad d\hat{\phi}_{ix} = d\phi_{ix}\ d\theta_{ix}, \tag{12}$$

$$\hat{J}_x(\hat{\phi}_{ix}) = J_x(\phi_{ix}) + \theta_{ix} J_x^B(\phi_{ix}). \tag{13}$$

The partition function (6) reads, in terms of the radial variables $\hat{\phi}_{ix}$,

$$\mathcal{Z}[\hat{J}] = \int \prod_x d^N \hat{\phi}_x e^{\hat{J}_x(\hat{\phi}_{ix})}\ \Delta(\phi_x)\ e^{\mathcal{W}[\hat{\phi}]} \tag{14}$$

$$\mathcal{W}[\hat{\phi}] = -\tfrac{1}{2} \sum_{x,x';i,j} C_{xx'} \ln|\phi_{ix} + \phi_{jx'} + \theta_{ix}\theta_{jx'}| \tag{15}$$

The partition function (14) generalizes the eigenvalue integral for the pure matrix theory introduced in [7]. Note that the pure matrix theory (no θ's) describing the closed string sector can be reformulated, using the Cauchy identity

$$\frac{\Delta(\phi)\Delta(\phi')}{\prod_{i,j}(\phi_i + \phi'_j)} = \det \frac{1}{\phi_i + \phi'_j}, \tag{16}$$

as a free Fermi system defined by a one-particle transfer matrix, much as the matrix quantum mechanics. However, after introducing the "quark" fields, the fermions of different colors start to interact. This is the main obstacle for generalizing the formalism of matrix quantum mechanics to open strings[2].

A field theory for the loop variables

The density $\hat{\rho}_x = \sigma_x + \theta\rho_x$ for the distribution of the radial coordinates $\hat{\phi}_{ix} = \{\phi_{ix}, \theta_{ix}\}$

$$\begin{aligned}\hat{\rho}_x(\phi,\theta) &= \delta(\phi - \Phi_x)\ \delta(\theta - \bar{\psi}_x\psi_x) = \sum_{i=1}^{N} \delta(\hat{\phi}, \hat{\phi}_{ix}) \\ &= \sum_{i=1}^{N} (\theta + \theta_{ix})\ \delta(\phi - \phi_{ix}) = \sigma_x(\phi) + \theta\rho_x(\phi).\end{aligned} \tag{17}$$

[2] This problem was have been considered originally by I. Affleck [8] and, more recently, by J. Minahan [9] and M. Douglas[10].

is the collective field for which the $1/N$ expansion makes sense of quasiclassical expansion. The Laplace transform of the density (ε is assumed to be a nilpotent variable as θ)

$$\hat{W}_x(\ell,\varepsilon) = W_x(\ell) + \varepsilon\Omega_x(\ell) = \int d\hat{\phi}\ e^{\ell\phi+\varepsilon\theta}\hat{\rho}_x(\phi,\theta). \tag{18}$$

gives the Wilson loop and line (5).

Let us perform a change of variables $\hat{\phi}_{ix} \ \rightarrow\ \hat{\rho}_x(\hat{\phi})$ in the integral (14). The action $\mathcal{W}$ becomes a quadratic form in $\hat{\rho}$

$$W[\ \hat{\rho}\ ] = \tfrac{1}{2}\hat{\rho}\cdot\hat{K}\cdot\hat{\rho} = \tfrac{1}{2}\rho\cdot K^{+}\cdot\rho + \tfrac{1}{2}\sigma\cdot K^{B}\cdot\sigma \tag{19}$$

where $\cdot$ stands for a sum and integral over the repeated variables and the kernel $\hat{K}$ reads explicitly

$$\hat{K}_{xx'}(\hat{\phi},\hat{\phi}') = -C_{xx'}\ln|\phi+\phi'+\theta\theta'| = K^{+}_{xx'}(\phi,\phi') + \theta\theta' K^{B}_{xx'}(\phi,\phi') \tag{20}$$

$$K^{+}_{xx'}(\phi,\phi') = -C_{xx'}\ln|\phi+\phi'|,\quad K^{B}_{xx'}(\phi,\phi') = -\frac{C_{xx'}}{\phi+\phi'}. \tag{21}$$

The Jacobian is expressed as usually as a functional integral over a Lagrange multiplier field[3] $\hat{\alpha}_x(\hat{\phi}) = \alpha_x(\phi) + \theta\beta_x(\phi)$,

$$\begin{aligned}\mathcal{J}[\hat{\rho}] &= \prod_x d^N\hat{\phi}_x\ \Delta(\phi_x)\int\mathcal{D}\hat{\alpha}\ \exp\Big[-\int d\hat{\phi}\hat{\alpha}_x(\hat{\phi})\big[\hat{\rho}_x(\hat{\phi}) - \sum_{i=1}^{N}\delta(\hat{\phi}-\hat{\phi}_{ix})\big]\big)\Big] \\ &= \int\mathcal{D}\hat{\alpha}_x e^{-\hat{\alpha}_x\cdot\hat{\rho}_x + \mathcal{F}_0[\alpha_x+\ln\beta_x]}\end{aligned} \tag{22}$$

where by $\mathcal{F}_0[V]$ we denoted the logarithm of the one-site integral in external field $-V(\phi)$

$$e^{\mathcal{F}_0[V]} = \int\prod_{i=1}^{N} d\phi_i\ e^{V(\phi_i)}\ \Delta^2(\phi)\ . \tag{23}$$

[3] In ref. [7] we have included the Vandermond determinant $\Delta(\phi_x)$ in the effective action. The resulting collective theory is not well defined at short distances and therefore ambiguous beyond the tree level. The healthy way to go to loop variables is to include the Vandermond determinant into the Jacobian $\mathcal{J}[\rho]$.

Combining (19) and (22) and shifting $\alpha \to \alpha - \ln\beta + J, \beta \to \beta + J_B$, we write the partition function (14) as the following functional integral

$$\mathcal{Z}[\hat{J}] = \int \mathcal{D}\hat{\rho}\mathcal{D}\hat{\alpha}\ \mathcal{W}_{\rm tot}[\hat{\rho}, \hat{\alpha}] \tag{24}$$

$$\mathcal{W}_{\rm tot}[\hat{\rho}, \hat{\alpha}] = \tfrac{1}{2}\hat{\rho} \cdot \hat{K} \cdot \hat{\rho} - \hat{\rho} \cdot \hat{\alpha} + \sum_x \Big(\mathcal{F}_0[\alpha_x + J_x] + \int d\phi \rho_x(\phi) \ln(\beta_x + J_x^B) \Big). \tag{25}$$

The string field theory will be obtained as the large-N quasiclassical expansion for this integral. For this purpose we have to solve the following technical problems: 1) find the classical string background $\hat{\rho}_c, \hat{\alpha}_c$, which is the solution of the saddle-point equations, 2) diagonalize the quadratic action, and 3) expand the interacting part[4] as a series in $1/N, \hat{\rho} - \hat{\rho}_c, \hat{\alpha} - \hat{\alpha}_c$. The solution of the first two problems is known (see refs. [5],[6]), and we will explain it without going into details.

Saddle point

The stationarity condition for the $\hat{\alpha}$-field gives

$$\rho_c = \left(\frac{\delta}{\delta \alpha} \mathcal{F}_0[\alpha + J] \right)_{\alpha = \alpha_c}, \quad \sigma_c = \frac{\rho_c}{\beta_c + J^B} \tag{26}$$

The first equation means that ρ_c coincides with the classical spectral density in the one-matrix integral, which is related to the potential $V = -\alpha_c$ by a linear equation. If we denote by K^- the linear operator with kernel ($\mathcal{P}$ means principal value prescription)

$$K^-_{xx'}(\phi, \phi') = -2\delta_{xx'} \mathcal{P} \ln|\phi - \phi'|, \tag{27}$$

then the first eq. (26) takes the form

$$K^- \cdot \rho_c = \alpha_c. \tag{28}$$

[4] We will treat the $1/N$-corrections to the tadpoles and propagators as interaction

Taking into account (26), (28) and neglecting the subleading term $\ln\beta$, we write the stationarity conditions for the $\hat{\rho}$-field as

$$(K^- - K^+)\cdot\rho_c \;=\; J, \qquad \frac{\rho_c}{\sigma_c} - K^B\cdot\sigma_c \;=\; J^B. \tag{29}$$

The first equation (29) determines the closed string background. It has been solves exactly for a stationary polynomial source J [11], [12] . In the scaling limit $\{\lambda\to\lambda^*, N\to\infty;\ N(\lambda^*-\lambda)=\Lambda\}$ the solution is supported by a semi-infinite interval

$$-\infty < \phi < -\sqrt{\mu} \tag{30}$$

and reads explicitly

$$\rho_c(\phi) = \sqrt{\phi^2-\mu}, \quad (1 << \mu << N). \tag{31}$$

The coupling constant κ for the string topogical expansion expansion is absorbed in the parameter μ. We can re-introduce it by the substitution $\phi\to\kappa^{-1/2}\phi, \mu\to\kappa^{-1}\mu$. Then each closed (open) string loop contributes a factor of κ^2 (κ). The renormalized string tension Λ is equal to μ up to logarithmic corrections typical for the $D=1$ string. There are two possible critical regimes characterized by different logarithmic violations ([5], [11]): $\Lambda\sim-\mu\ln\mu$ (dilute critical regime) , $\Lambda\sim\mu[\ln\mu]^2$ (dense critical regime). The choice of the source stemming from the action in (3) will lead to the dense critical regime. The dilute regime is obtained by introducing another coupling and tuning it.

The second nonlinear equation (29), which determines the open string background, depends on the closed string background and on a second parameter, the renormalized mass $\mu_B\sim(\lambda^*_B-\lambda_B)\sqrt{N}$ of the ends of the open string. Its general solution has been found in [6]. Here we will restrict ourselves to the the case of vanishing "quark" mass $\mu_B=0$. In this case the solution is given by

$$\sigma_c = \frac{1}{\sqrt{2\pi}}(|\phi|-\sqrt{\mu})^{1/2}, \quad \beta_c + J^B = \sqrt{2\pi}(|\phi|+\sqrt{\mu})^{1/2}. \tag{32}$$

Diagonalization of the quadratic action

To study the string excitations we shift the fields by their classical values. It is also convenient to parametrize the eigenvalue interval by the "time-of-flight" variable τ ranging from 0 to ∞

$$\tau = -\int^{\phi} \frac{d\phi}{\rho_c(\phi)} \ , \qquad \phi(\tau) = -\sqrt{\mu} \cosh \tau \tag{33}$$

and make the following redefinition of the fields

$$\rho - \rho_c = \partial_\phi \chi = -\frac{\partial_\tau \chi}{\rho_c}, \qquad \alpha - \alpha_c = K^- \cdot \partial_\phi \tilde{\chi} = -K^- \cdot \frac{\partial_\tau \tilde{\chi}}{\rho_c} \tag{34}$$

$$\frac{\sigma}{\sigma_c} = 1 - \frac{\psi}{\rho_c}, \quad \frac{\beta + J^B}{\beta_c + J^B} = 1 - \frac{\tilde{\psi}}{\rho_c} \tag{35}$$

where the new fields are considered as functions of x and τ. The quantum parts of the loop fields (18) are related to the fields in the τ-space by

$$W_x(\ell) = \int_0^\infty d\tau \ e^{-\sqrt{\mu}\cosh \tau} \partial_\tau \chi(x,\tau); \ \ \Omega_x(\ell) = \int_0^\infty d\tau \ e^{-\sqrt{\mu}\cosh\tau} \psi(x,\tau). \tag{36}$$

The operators $K^{\pm}$ and K^B are now represented by the kernels

$$\begin{aligned}
\mathcal{K}^+_{xx'}(\tau,\tau') &= -C_{xx'} \partial_\tau \partial_{\tau'} \ln|\phi + \phi')|, \\
\mathcal{K}^-_{xx'}(\tau,\tau') &= -2\delta_{xx'} \ \partial_\tau \partial_{\tau'} \mathcal{P} \ln|\phi - \phi'|, \\
\mathcal{K}^B_{xx'}(\tau,\tau') &= C_{xx'} \frac{\sigma_c(\phi)\sigma_c(\phi')}{|\phi+\phi'|}.
\end{aligned} \tag{37}$$

where integration is assumed to go in the interval $0 < \tau < \infty$. It is easy to see that for the particular background (31), (32), in which $\phi = -\sqrt{\mu}\cosh \tau$, these kernels are diagonalized by plane waves

$$\langle E, p | \tau, x \rangle = \frac{1}{\sqrt{\pi}} \sin E\tau e^{i\pi p x}. \tag{38}$$

If x is considered as a continuous variable, then these kernels represent the following finite-difference operators

$$\mathcal{K}^+ = \frac{2\pi \ \partial_\tau \cosh \partial_x}{\sin \pi \partial_\tau}, \ \mathcal{K}^- = \frac{2\pi \ \partial_\tau \cos \pi \partial_\tau}{\sin \pi \partial_\tau}, \ \mathcal{K}^B = \frac{\cosh \partial_x}{\cos \pi \partial_\tau}. \tag{39}$$

To write the quadratic action we need the second term of the Taylor expansion of the functional $\mathcal{F}_0$ around α_c. This term is equal to $\frac{1}{2}\tilde{\chi}\cdot\mathcal{K}^-\cdot\tilde{\chi}$ because the new field $\tilde{\chi}$ is the fluctuating part of the spectral density in the one-matrix integral. The quadratic action takes the form, in terms of the new fields[5]

$$\mathcal{W}^{\rm free}=\tfrac{1}{2}\chi\mathcal{K}^+\chi+\tfrac{1}{2}\tilde{\chi}\mathcal{K}^-(\tilde{\chi}-2\chi)+\tfrac{1}{2}\psi\mathcal{K}^B\psi+\tfrac{1}{2}\tilde{\psi}(\tilde{\psi}-2\psi). \tag{40}$$

A complete diagonalization is achieved if we introduce the ghost-like fields

$$\chi^{(1/2)}=\tilde{\chi}-\chi,\quad \psi^{(1/2)}=\tilde{\psi}-\psi \tag{41}$$

which decouple from χ,ψ,

$$\mathcal{W}^{\rm free}=\tfrac{1}{2}\chi(\mathcal{K}^+-\mathcal{K}^-)\chi+\tfrac{1}{2}\psi(\mathcal{K}^B-1)\psi+\tfrac{1}{2}\chi^{(1/2)}\mathcal{K}^-\chi^{(1/2)}+\tfrac{1}{2}\psi^{(1/2)}\psi^{(1/2)}. \tag{42}$$

The effect of the (1/2)-fields is that the internal propagators are modified by subtracting their values at $p=1/2$. Note that the term to be subtracted from the closed string propagator coincides with the loop-loop correlator in the $D=0$ string theory. This can be expected, since the expansion around the mean field (the solution of the one-matrix integral) is in some sense expansion around the string theory without embedding. It is possible to make the subtraction at another point but not to eliminate it by a redefinition of the vertices. Without such a subtraction the E-integration would produce singularities when calculating loops.

We see that both closed and open strings have the same spectrum of on-shell states $iE=\pm p+2n,\ n\in\mathbb{Z}$, that forms the light cone in a Minkowski space (iE,p) with periodic momentum coordinate. Each on-shell state creates a "microscopic loop" on the world sheet with given momentum p and corresponds to a local scaling operator. It can be thought of as a product of a vertex operator (the state with minimal energy $E=|p|$) and local operators representing infinitesimal deformations of the microscopic loop. The states with given momentum p form an infinite tower of "gravitational descendants" of this vertex operator.

[5] The quadratic action for the $D=0$ string theory was diagonalized in ℓ-space by Moore, Seiberg and Staudacher [13]. A subtle point is that the eigenvalues of the same operator acting in ℓ-space and in τ-space differ by a factor $\Gamma(iE)\Gamma(-iE)$. This is possible because the two spaces are related by a nonunitary transformation. For details see [14].

Interactions

The interaction part of the action (25) consists of all terms that disappear in the planar limit. Thus we treat as interaction the nonplanar corrections to the tadpoles and the quadratic term mixing the open and closed string fields. In terms of the new fields (34) the interaction potential reads

$$\begin{aligned} \mathcal{W}^{\rm int} &= \sum_x \left\{ \mathcal{F}_0[\alpha_c + K^- \cdot \partial_\phi \tilde{\chi}_x] - \int d\tau (\rho_c^2 - \partial_\tau \chi) \ln(1 - \ \tilde{\psi}/\rho_c) \right\}_< \\ &= \mathcal{U}^{\rm closed}[\tilde{\chi}] + \mathcal{U}_{\rm open}[\tilde{\psi}] + \mathcal{U}^{\rm closed}_{\rm open}[\chi, \tilde{\psi}] \end{aligned} \tag{43}$$

where $\{ \ \}_<$ means that only the negative powers of μ are retained. The individual terms in the expansion of (43) in the fields and in $1/\mu$ can be associated with surfaces with negative global curvature, localized at the sites $x \in \mathbb{Z}$.

To fix the form of the closed string vertices we need to know the Taylor expansion

$$\mathcal{F}_0[\alpha_c + \alpha_x] - \mathcal{F}_0[\alpha_c] = \sum_{n=1}^{\infty} \frac{1}{n!} \int d^n\phi \ A_n \cdot \alpha_x^{\otimes n}. \tag{44}$$

The coefficient functions $A_n(\phi_1, ..., \phi_n)$ are the n-point correlation functions for the spectral density in the one-matrix integral with potential $V(\phi) = -\alpha_c(\phi)$, and can be obtained as the discontinuities of the n-loop correlators. A closed expression for the tree-level loop amplitudes for an arbitrary potential was found by Ambjorn, Jurkiewicz and Makeenko [15]. The problem is not yet completely solved but the general form of the loop amplitudes beyond the tree level is known [5], [16]. We have, for the potential $V(\phi) = -\alpha_c(\phi) = (2/\pi)\tau \sinh \tau$,

$$A_n(\phi_1, ..., \phi_n) \prod_{k=1}^{n} \frac{\partial \phi_k}{\partial \tau_k} = \mu^{2-n} \left[\mathcal{A}_n\Big(\frac{1}{\mu}, \frac{\partial}{\partial a}\Big) \ \frac{\partial^{n-3}}{\partial a^{n-3}} \ \prod_{k=1}^{n} \frac{\partial}{\partial \tau_k} \sin\Big(\frac{\partial}{\partial \tau_k}\Big) \frac{1}{\sqrt{\cosh \tau_k + a}} \right]_{a=1} \tag{45}$$

where the function $\mathcal{A}_n$ is defined as the formal series

$$\mathcal{A}_n(x, y) = \sum_{h=0}^{\infty} \sum_{k=0}^{3h} \mathcal{A}_n^{(h,k)} \ x^{2h} \ y^k. \tag{46}$$

where the 3 coefficients with $n + 2h - 2 \leq 0$ are assumed equal to zero. The coefficient $A_n^{(h,k)}$ can be associated with a sphere with n boundaries and h handles. The origin of the factors $\sin(\partial/\partial\tau_k)$ is that the correlation functions for the spectral density are equal to the discontinuities of the loop amplitudes as functions of the complex variables $z_k = \sqrt{\mu}\cosh\tau_k$ along the cuts $-\infty < z_k < -\sqrt{\mu}$. of Using the operator representation (39) we see that the potential (43) depends on the field $\tilde{\chi}$ through a discrete set of projections Π_n whose generating function is given by

$$\Pi(a)\tilde{\chi} \equiv \frac{\pi}{\sqrt{2}} \sum_n \left(\frac{a-1}{2}\right)^n \Pi_n \tilde{\chi} = \int_0^\infty \frac{d\tau}{\sqrt{\cosh\tau + a}} \partial_\tau \ \cos\pi\partial_\tau \ \tilde{\chi}(\tau, x). \tag{47}$$

By Fourier-transforming and using the identity

$$\sqrt{2}\frac{\cosh\pi E}{\pi} \int_0^\infty d\tau \frac{\cos E\tau}{\sqrt{\cosh\tau + a}} = P_{-\frac{1}{2}+iE}(a) = \sum_{n=0}^\infty \left(\frac{1-a}{2}\right)^n \frac{(\frac{1}{2}+iE)_n(\frac{1}{2}-iE)_n}{n!\ n!} \tag{48}$$

we find the explicit expression of (47) in terms of the derivatives of $\tilde{\chi}$ at the point $\tau = 0$

$$\Pi_n\tilde{\chi} = \frac{1}{(n!)^2}\left(\partial_\tau \prod_{j=0}^{n-1}[(j+\tfrac{1}{2})^2 - \partial_\tau^2]\tilde{\chi}(\tau, x)\right)_{\tau=0} \tag{49}$$

Thus the potential for the closed string interactions depends on the field $\tilde{\chi}$ only through its normal derivatives $\partial_\tau^{2n+1}\chi(\tau, x), n = 0, 1, ...,$ along the edge $\tau = 0$ of the half-plane

$$\mathcal{U}_{\text{closed}}(\tilde{\chi}) = \sum_x \sum_{n=1}^\infty \mu^{2-n}\left[\frac{\partial^{n-3}}{\partial a^{n-3}} A_n\left(\frac{1}{\mu}, \frac{\partial}{\partial a}\right) \frac{[\Pi(a)\tilde{\chi}]^n}{n!}\right]_{a=1} \tag{50}$$

The interaction of closed strings is nonzero only along the wall $\tau = 0$, which qualitatively in accord with the collective theory for strings with continuum target space [17], [18].

The potential for the interaction of open strings consists of two terms. The first term describes interaction involving only open strings

$$\mathcal{U}_{\text{open}}[\tilde{\psi}] = \sum_{n=3}^\infty \frac{\mu^{1-n/2}}{n} \sum_x \int d\tau[\sinh\tau]^{2-n}[\tilde{\psi}(x, \tau)]^n \tag{51}$$

and its n-th term has the geometrical meaning of a disc where n strips meet. The factor $\mu^{1-n/2}$ is associated with the geodesic curvature of the pieces of boundary separating the

strips. Contrary to the closed string, the potential for open string interactions is smooth and only exponentially decaying in the bulk. The second term

$$\mathcal{U}_{\rm open}^{\rm closed} = \sum_{n=1}^{\infty} \frac{1}{n} \mu^{-n/2} \sum_{x} \int d\tau (\sinh \tau)^{-n} \chi(x,\tau) \partial_\tau [\tilde{\psi}(x,\tau)]^n \tag{52}$$

describes the interaction one open string and a number of open strings. The n-th term has the geometrical meaning of a surface with the topology of a cylinder connecting one tube and n strips. The lowest vertex ($n = 1$) describes the transition between one open and one closed string state. The nonplanar corrections to the open string tadpole are are composed from one such vertex and a nonplanar closed-string tadpole, etc.

In conclusion, we have constructed, up to some numerical factors, the complete interacting potential for the field theory of closed and open strings with discrete target space. We believe that this potential describes as well the interactions of closed and open strings with target space $\mathbb{R}$. The tree-level dynamics in the open-string sector following from the potential (51) is in qualitative agreement with the amplitudes obtained by Bershadsky and Kutasov [19] from the Liouville theory. Moreover, their open string amplitudes follow from an effective lattice model very similar to our collective theory in the τ-parametrization.

Here we considered only the case of massles fermions, $\mu_B = 0$. If $\mu_B \neq 0$, then the open string background is asymptotically approaching (32) when $\tau \to \infty$, the deviation being exponentially small in τ. This weak dependence on τ will affect the interaction potential but not the the spectrum of the open string excitations (for details see [6]). In the same way the spectrum of the closed string does not depend on the string tension μ. This stability of the spectrum is the major discrepancy between the bosonic string and the strings expected to describe the dynamics of flux tubes in QCD.

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